Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2019

Subject Name : Advanced Functional Analysis
Subject Code :5SC04AFA1 **Branch: M.Sc. (Mathematics)**
Semester : 4 **Date : 15/04/2019** **Time : 02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

- **a.** Let $\{x, y\}$ be an orthonormal set in a Hilbert space H. Then find the value of $\|x+y\|^2$.
- **b.** Let X be an inner product over $\mathbb{C}$. If $x, y \in X$, then find the value of $\langle x+y, x+ y-x-y,x-y+ix+iy,x+iy-i\langle x-iy,x-iy\rangle$.
- **c.** Let H be a Hilbert space, and let $T \in BL(H)$ be one one. Define $\langle\cdot,\cdot\rangle_T$ by $\langle x, y\rangle_T = \langle Tx, Ty\rangle$ for all $x, y \in H$. Show that $\langle\cdot,\cdot\rangle_T$ is an inner product on H.
- **d.** Let X be an inner product space, and let $x_0 \in X$. Define f on X by $f(x) = \langle x, x_0\rangle$ for all $x \in X$. Show that f is a linear map.
- **e.** Give an example of a weakly convergent sequence in some Hilbert space which is not convergent.
- **f.** Define best approximation.
- **g.** Give an example of a normed space which is not separable.

Q-2 **Attempt all questions** **(14)**

(a) Let X be an inner product space, and let E be an orthonormal subset of X. If $x \in X$, then show that the set $E_x = \{y \in E : \langle x, y\rangle \neq 0\}$ is countable.

(b) If H is an infinite dimensional Hilbert space with a countable orthonormal basis, then prove that H is isometrically isomorphic to ℓ^2.

OR

Q-2 **Attempt all questions** **(14)**

(a) Show that the normed linear space $(\ell^p, \|\cdot\|_p)$ is an inner product space if and only if $p = 2$.

(b) Let X be an inner product space, and $x, y \in X$. Show that $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$. Also, prove that equality holds if and only if x and y are linearly dependent.

Q-3 **Attempt all questions** **(14)**

(a) Let H be a Hilbert space. Show that a linear map $f : H \to \mathbb{C}$ is continuous if and only if there is unique $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$.

(b) Let X be an inner product space, Y be a subspace of X and $x \in X$. Prove that $y \in Y$ is a best approximation from Y to x if and only if $(x - y) \perp Y$.

OR

Q-3 (a) Let H be a Hilbert space, and let Y be a closed subspace of H. If g is a continuous linear functional on Y, then prove that there is unique continuous linear functional f on H such that $f_{|Y} = g$ and $\|f\| = \|g\|$.

(b) If $E \neq \emptyset$ is a closed convex subset of a Hilbert space H, then show that E contains a unique element of minimum norm.

SECTION – II

Q-4 Attempt the Following questions (07)

- **a.** If $T \in BL(H)$, then show that $\ker T = \ker(T^*T)$.
- **b.** Let $T \in BL(H)$. Show that$\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$.
- **c.** If $H$is a complex Hilbert space and if $T \in BL(H)$, then show that there exist self-adjoint bounded linear maps A, B such that $T = A + iB$.
- **d.** If $T \in BL(H)$ is compact and $S \in BL(H)$, then show that TS is compact.
- **e.** Let $T: \ell^2 \to \ell^2$be $T(x(1), x(2), \dots) = (x(2), x(3), \dots)$. Show that T is not an isometry.
- **f.** Let $T \in BL(H)$ be normal. If λ and μ are distinct eigen values of T, then show that the corresponding eigen vectors are orthogonal.
- **g.** If $T \in BL(H)$ is bounded below, then show that it is one one.

Q-5 Attempt all questions (14)

(a) Let H be a Hilbert space. If $T \in BL(H)$ is self-adjoint, then show that $\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leq 1\}$.

(b) Let Hbe a Hilbert space. If $T \in BL(H)$, then show that there is unique $T^* \in BL(H)$ such that $\langle x, T^*y\rangle = \langle Tx, y\rangle$ for all $x, y \in H$ and $\|T^*\| = \|T\|$.

OR

Q-5 (a) Let H be a Hilbert space, and let $T \in BL(H)$ be onto. Prove that T^* is bounded below.

(b) Let H be a Hilbert space and $T \in BL(H)$. Prove that T is unitary if and only if T is an onto isometry.

Q-6 Attempt all questions (14)

(a) Let H be a Hilbert space, and let $T \in BL(H)$ be compact. If $0 \neq \lambda \in \sigma_a(T)$, then show that $\lambda \in \sigma_e(T)$ and $\ker(T - \lambda I)$ is finite-dimensional.

(b) Let H be a Hilbert space, $H \neq \{0\}$ and $T \in BL(H)$. If T is self-adjoint, then show that $m_T \in \sigma(T)$, where $m_T = \inf\{\lambda : \lambda \in W(T)\}$.

OR

Q-6 Attempt all Questions

(a) Let H be a Hilbert space and $T \in BL(H)$. Prove that $\sigma(T) = \sigma_a(T) \cup \{\bar{\mu} : \mu \in \sigma_e(T^*)\}$.

(b) Let H be a Hilbert space. If $T: H \to H$ is a compact linear map, then show that $T \in BL(H)$. Is the converse true? Justify.